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SEPARATED FLOW MODELS—II

HIGHER ORDER DISPERSION EFFECTS IN THE AVERAGED FORMULATION

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Abstract—In a previous paper (Part I), it was shown that the averaged formulation for stratified flow did not appear to contain the higher order dispersion terms that were obtained on analysing the local instantaneous two-dimensional formulation. In this paper, this apparent inconsistency is resolved by more careful modelling of the difference between the phase average and interfacial pressures. The resulting set of averaged conservation equations are shown to have the correct linear dispersion relationship for long waves. Asymptotic analysis of these averaged equations by the method of reductive perturbation also leads to description of finite amplitude waves by a Koretweg–de Vries equation that is identical to that obtained previously from the local instantaneous formulation.

1. INTRODUCTION

In many two-phase flow problems, averaged values of parameters, like void fraction, are of engineering interest. Therefore, one of the main approaches to two-phase flow modelling has been to use averaged versions of the local instantaneous conservation equations. Separate sets of averaged conservation equations are usually derived for each phase, or the relatively homogeneous portions of each phase, and these are then coupled by interfacial transfer relationships and jump conditions. "Multifluid" models of this type have been discussed by several investigations (e.g. Saito 1977, Delhaye & Achard 1976, Wallis 1976).

While averaged models are of considerable practical importance, it is clear that information is lost in the averaging process and must be supplied in the form of essentially empirical auxiliary relationships. To investigate some of the effects of averaging, Part I of this paper (Banerjee & Chan 1979) considered wave propagation and dispersion in stratified flow using both the averaged and local instantaneous formulations. It was found that the local instantaneous conservations equations led to a description of finite amplitude waves by the non-linear Korteweg–de Vries equations (Jeffrey & Kakutani 1972) which have third derivative dispersion terms. The averaged model, however, led to purely hyperbolic waves for the same physical situation. It is known (Jeffrey 1967) that even with the analytic initial data, solutions of hyperbolic equations will steepen in time and eventually become multivalued. On the other hand, if higher order derivative terms are present, albeit with very small coefficients, then these eventually balance the nonlinear steepening and can, in some situations, lead to waves of permanent shape (solitary waves). Analysis of systems containing higher order dispersive terms reveal a richness of phenomena, such as the remarkable results regarding the interaction of solitons (Whitham 1974) that is by no means possible for purely hyperbolic waves.

For these reasons, it was considered worthwhile to re-examine the averaged formulation to determine whether more careful modelling could lead to results regarding wave phenomena consistent with those from the local instantaneous equations. In order to use some of the derivations from Part I, the analysis was initially done for stratified two-phase in horizontal ducts.

2. AVERAGED CONSERVATION EQUATIONS FOR STRATIFIED FLOW

Consider the stratified flow of two incompressible fluids in a horizontal duct of height H as shown in figure 1. To compare the results with those obtained in Part I, we will assume inviscid flow with no interphase mass transfer. The averaged conservation equations are (see Part I):

Mass

$$\frac{\partial \alpha_k}{\partial t} + u_k \frac{\partial \alpha_k}{\partial z} + \alpha_k \frac{\partial u_k}{\partial z} = 0. \qquad [1]$$

Momentum

$$\alpha_k \rho_k \left(\frac{\partial u_k}{\partial t} + u_k \frac{\partial u_k}{\partial z} \right) + \alpha_k \frac{\partial p_k}{\partial z} - (p_i - \langle p_k \rangle) \frac{\partial \alpha_k}{\partial z} = 0. \qquad [2]$$

Where α, u, ρ and p are phase volume fraction, axial velocity, density and pressure respectively. The subscript k refers to the phase (liquid or gas) and the subscript i to the interface. The time coordinate is t and the coordinate in the flow direction is z. We have written

$$u_k = \overline{\langle u_k \rangle}_k = \overline{\alpha_k \langle u_k \rangle} / \overline{\alpha_k} \,,$$

and

$$C_k = \overline{\alpha_k \langle u_k^2 \rangle} / (\overline{\alpha_k} \overline{\langle u_k \rangle}_k^2) = 1.0 \,.$$

The averaging signs have been eliminated from [1] and [2] for all terms except $\langle p_k \rangle$ (which is kept to avoid confusion later on).

In the previous paper (Part I), the difference between the interface and phase average pressures was taken to be

$$p_i - \langle p_g \rangle = \frac{\rho_g g \alpha H}{2}, \qquad [3]$$

and

$$p_i - \langle p_l \rangle = -\frac{\rho_l g (1-\alpha) H}{2}, \qquad [4]$$

where g is the gravitational constant and H is the duct height.

This was a static approximation, because the general case would involve derivatives of the cross stream velocity. To improve the model given by [1]–[4], it is necessary to consider the transverse momentum equation for each phase, which is

$$-\frac{\partial p_k}{\partial y} = \rho_k \left[\frac{\partial v_k}{\partial t} + u_k \frac{\partial v_k}{\partial z} \right] + \rho_k g \,, \qquad [5]$$

where v_k is the cross stream velocity of phase k and y is the cross stream direction.

The cross stream velocity derivative is neglected, and inviscid flow with no mass transfer is assumed.

To find the average phasic pressure $\langle p_k \rangle$, [5] must be integrated with respect to y. This is straightforward if the term in parenthesis on the r.h.s. is not considered, when the average phasic pressures are given by [3] and [4]. However, if the dynamic terms are retained then their functional dependence on y must be found. To do this rigorously, the full two-dimensional equations must be solved.

However, in this paper we will make approximations that will simplify the problem considerably, but are expected to apply reasonably well to long waves only. Let

$$\xi_k \stackrel{\Delta}{=} \rho_k\left[\frac{\partial v_k}{\partial t} + u_k \frac{\partial v_k}{\partial z}\right]. \qquad [6]$$

Then we have the boundary conditions

$$\xi_k = \xi_{ki} \quad \text{at} \quad y = Y,$$

$$\xi_g = \frac{\partial \xi_g}{\partial y} = 0 \quad \text{at} \quad y = H,$$

$$\xi_l = \frac{\partial \xi_l}{\partial y} = 0 \quad \text{at} \quad y = 0. \qquad [7]$$

The derivations vanish at the walls because $\partial v_k/\partial y = 0$ from continuity considerations. The simplest polynomial function of y that will fit these boundary conditions is a quadratic. Clearly this is a poor approximation for short waves when points of inflection may occur; however, it is probably reasonable for long waves. Integrating [5] with a quadratic in y for ξ_k and substituting the boundary conditions [7] we obtain

$$\langle p_g\rangle - p_i = -\left[\frac{\rho_g g\alpha H}{2} + \frac{1}{3}\xi_{gi}\alpha H\right], \qquad [8]$$

$$\langle p_l\rangle - p_i = \left[\frac{\rho_g g(1-\alpha)H}{2} + \frac{1}{3}\xi_{li}(1-\alpha)H\right]. \qquad [9]$$

Note the additional terms that have appeared in comparison with [3] and [4]. To proceed we use the kinematic condition

$$v_{gi} = \frac{\partial Y}{\partial t} + u_g \frac{\partial Y}{\partial z}, \qquad [10]$$

where Y is the interface position.

It follows from [10] that

$$\xi_{gi} = -\rho_g H\left[\frac{\partial}{\partial t} + u_g \frac{\partial}{\partial z}\right]^2 \alpha, \qquad [11]$$

and similarly

$$\xi_{li} = -\rho_l H\left[\left(\frac{\partial}{\partial t} + u_l \frac{\partial}{\partial z}\right)^2 \alpha\right]. \qquad [12]$$

We may now substitute [8], [9], [11] and [12] into the phasic momentum equations [2] to obtain the momentum equations

$$\alpha\rho_g\left[\frac{\partial u_g}{\partial t} + u_g \frac{\partial u_g}{\partial z}\right] + \alpha\frac{\partial p_i}{\partial z} - \rho_g g\alpha H\frac{\partial \alpha}{\partial z} + \frac{1}{3}\rho_g\alpha^2 H^2\left\{\frac{\partial}{\partial z}\left(\frac{\partial}{\partial t} + u_g\frac{\partial}{\partial z}\right)^2\alpha\right\} = 0, \qquad [13]$$

and

$$(1-\alpha)\rho_l\left[\frac{\partial u_l}{\partial t} + u_l\frac{\partial u_l}{\partial z}\right] + (1-\alpha)\frac{\partial p_i}{\partial z} - \rho_l g(1-\alpha)H\frac{\partial \alpha}{\partial z} - \frac{1}{3}\rho_l(1-\alpha)^2 H^2$$

$$\times\left\{\frac{\partial}{\partial z}\left(\frac{\partial}{\partial t} + u_l\frac{\partial}{\partial z}\right)^2\alpha\right\} = 0. \qquad [14]$$

In obtaining these equations we have neglected terms of the type $\xi_{ki}(\partial\alpha/\partial z)$ since they enter neither in the linear dispersion analysis presented in the next section, nor in the subsequent reductive perturbation analysis, to the order considered, for finite amplitude waves. The phasic momentum equations [13] and [14] are the same as obtained previously in Part I, except for the last term on the l.h.s. of each equation.

3. LINEAR DISPERSION ANALYSIS

If $\bar{U}$ is the vector of dependent variables u_g, u_l, p_i and α, consider a perturbation of the form $\bar{U}\exp[i(kz - \omega t)]$. When the conservation equations [1], [13] and [14] are perturbed in this way, we obtain

$$\begin{bmatrix} \alpha\rho_g[u_g-\lambda] & & \alpha & \begin{matrix} -\rho_g g\alpha H \\ +\frac{1}{3}\alpha^2\rho_g H^2(\omega - u_g k)^2 \end{matrix} \\ & (1-\alpha)[u_l-\lambda] & 1-\alpha & \begin{matrix} -\rho_l g(1-\alpha)H \\ + \\ \frac{1}{3}(1-\alpha)^2\rho_l H^2(\omega - u_l k)^2 \end{matrix} \\ \alpha\rho_g & & & \rho_g[u_g-\lambda] \\ & -(1-\alpha)\rho_l & & \rho_l[u_l-\lambda] \end{bmatrix} \begin{bmatrix} \hat{u}_g \\ \hat{u}_k \\ \hat{p}_i \\ \hat{\alpha} \end{bmatrix} = 0, \qquad [15]$$

where the equations have been divided by ik and $\omega/k = \lambda$.

For a non-trivial solution, the determinant of the matrix must vanish, leading to the dispersion relationship

$$\alpha\rho_l(\lambda - u_l)^2\left[1 + \frac{(1-\alpha)^2k^2H^2}{3}\right] + (1-\alpha)\rho_g(\lambda - u_g)^2\left[1 + \frac{\alpha^2k^2H^2}{3}\right]$$
$$-\alpha(1-\alpha)gH(\rho_l - \rho_g) = 0, \qquad [16]$$

This dispersion relationship is similar to that obtained in Part I, except that additional terms involving k^2 now appear.

To determine whether [16] is correct, a comparison with the form in Milne–Thomson (1960) based on analysis of the two-dimensional local instantaneous equations is necessary. Milne–Thomson derives the dispersion relationship for the stratified flow of two incompressible inviscid fluids as

$$\rho_g(\omega - u_g k)^2\coth(kh) + \rho_l(\omega - u_l k)^2\coth[k(H-h)] - kg(\rho_l - \rho_g) = 0. \qquad [17]$$

The symbols are defined in figure 1.

Now

$$\coth(a) = \frac{1}{a} + \frac{a}{3} + \cdots. \qquad [18]$$

For small values of a, retaining the first two terms is sufficient. Therefore [17] simplifies to

$$\frac{\rho_g}{h}\left[\frac{\omega}{k} - u_g\right]^2\left[1 + \frac{k^2h^2}{3}\right] + \frac{\rho_l}{H-h}\left[\frac{\omega}{k} - u_l\right]^2\left[1 + \frac{k^2(H-h)^2}{3}\right] - g(\rho_l - \rho_g) = 0. \qquad [19]$$

Noting that $(h/H) = \alpha$ and $(H-h)/H = 1-\alpha$, we obtain a form identical to [16]. This suggests that the linear dispersion analysis obtained from the averaged conservation equations is correct

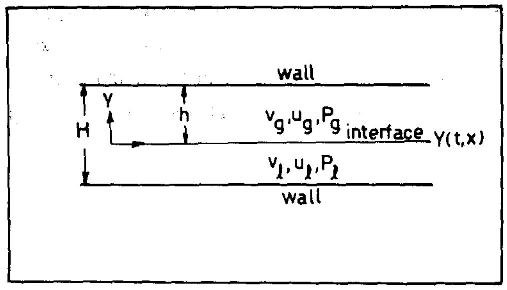

Figure 1. Definition of symbols and geometry for analysis of stratified flow.

for long waves, i.e. small values of kh and $k(H-h)$. This restriction is to be expected because of the approximations made in modelling the difference between the phase average and interfacial pressure.

The interesting conclusion from the linear dispersion relationship [16] is that the system of conservation equations [1], [13] and [14] do not lead to purely hyperbolic waves, but now lead to a dispersive system even in the absence of frictional, surface tension and mass transfer effects.

It is clear that the dispersion relationship [16] can be expanded in the form

$$\omega = \hat{a}k + \hat{b}k^3 + \cdots \qquad [20]$$

where $\hat{a}$ and $\hat{b}$ are real constants, which suggests that the weak nonlinearity may be considered by applying the reductive perturbation method as applied to hydromagnetic waves in a collision free plasma by Kakutani *et al.* (1968). The results of such an analysis are presented in the next section.

4. ASYMPTOTIC ANALYSIS FOR FINITE AMPLITUDE WAVES

Linear dispersion relationships of the type in [20] will often lead to the Korteweg–deVries equation for small, but finite amplitude, long waves. This is not always true, as shown by Kakutani & Ono (1969) for the example of Alfven waves which are described by a modified Korteweg–deVries equation.

In investigating finite amplitude waves we will use the procedure of semicharacteristic coordinate stretching and asymptotic expansions for the dependent variables suggested by Gardner & Morikawa (1960). The asymptotic solution does not, of course, contain all the properties of the original problem, but this is usually more than compensated for by the insight gained from the properties of the results obtained. Consider expansions of the dependent variables of the form

$$\begin{aligned} u_k &= u_k^{(0)} + \epsilon u_k^{(1)} + \epsilon^2 u_k^{(2)} + \cdots \\ p_i &= p_i^{(0)} + \epsilon p_i^{(1)} + \epsilon^2 p_i^{(2)} + \cdots \\ \alpha &= \alpha^{(0)} + \epsilon \alpha^{(1)} + \epsilon^2 \alpha^{(2)} + \cdots. \end{aligned} \qquad [21]$$

The small parameter $\epsilon = O(k^2)$ measures the weakness of dispersion.

We will use the semicharacteristic stretching transformation (Gardner & Morikawa 1960)

$$\tau = \epsilon^{3/2} t \quad \text{and} \quad \xi = \epsilon^{1/2}(x/V_0 - t), \qquad [22]$$

where V_0 is the phase velocity equivalent to $\hat{a}$ in the dispersion relationship [20] and ϵ is a small parameter of the order of the amplitude of the disturbances.

We will also assume that the flow is in a uniform state upstream at infinity, and therefore have the boundary conditions

$$
\begin{aligned}
&u_k^{(j)} \to 0 \\
&p_i^{(j)} \to 0 \qquad \text{as} \quad \xi \to -\infty \\
&\alpha^{(j)} \to 0
\end{aligned} \qquad [23]
$$

for $j = 1, 2, \ldots$.

When the expansions from [21] are substituted into the conservation equations [1], [13] and [14], and the coordinate transformation [22] is applied we obtain:

for $\epsilon^{3/2}$

$$\left(\frac{u_g^{(0)}}{V_0} - 1\right)\frac{\partial \alpha^{(1)}}{\partial \xi} + \frac{\alpha^{(0)}}{V_0}\frac{\partial u_g^{(1)}}{\partial \xi} = 0, \qquad [24]$$

$$\left(\frac{u_l^{(0)}}{V_0} - 1\right)\frac{\partial \alpha^{(1)}}{\partial \xi} + \frac{(1-\alpha^{(0)})}{V_0}\frac{\partial u_l^{(1)}}{\partial \xi} = 0, \qquad [25]$$

$$\left(\frac{u_g^{(0)}}{V_0} - 1\right)\frac{\partial u_g^{(1)}}{\partial \xi} + \frac{1}{\rho_g V_0}\frac{\partial p_i^{(1)}}{\partial \xi} - \frac{gH}{V_0}\frac{\partial \alpha^{(1)}}{\partial \xi} = 0, \qquad [26]$$

$$\left(\frac{u_l^{(0)}}{V_0} - 1\right)\frac{\partial u_l^{(1)}}{\partial \xi} + \frac{1}{\rho_l V_0}\frac{\partial p_i^{(1)}}{\partial \xi} - \frac{gH}{V_0}\frac{\partial \alpha^{(1)}}{\partial \xi} = 0, \qquad [27]$$

and

for $\epsilon^{5/2}$

$$\left(\frac{u_g^{(0)}}{V_0} - 1\right)\frac{\partial \alpha^{(2)}}{\partial \xi} + \frac{\alpha^{(0)}}{V^{(0)}}\frac{\partial u_g^{(2)}}{\partial \xi} = -\frac{u_g^{(1)}}{V_0}\frac{\partial \alpha^{(1)}}{\partial \xi} - \frac{\alpha^{(1)}}{V_0}\frac{\partial u_g^{(1)}}{\partial \xi} - \frac{\partial \alpha^{(1)}}{\partial \tau}, \qquad [28]$$

$$\left(\frac{u_l^{(0)}}{V_0} - 1\right)\frac{\partial \alpha^{(2)}}{\partial \xi} + \frac{1-\alpha^{(0)}}{V_0}\frac{\partial u_l^{(2)}}{\partial \xi} = -\frac{u_l^{(1)}}{V_0}\frac{\partial \alpha^{(1)}}{\partial \xi} - \frac{\alpha^{(1)}}{V_0}\frac{\partial u_l^{(1)}}{\partial \xi} - \frac{\partial \alpha^{(1)}}{\partial \tau}, \qquad [29]$$

$$
\begin{aligned}
\left(\frac{u_g^{(0)}}{V_0} - 1\right)\frac{\partial u_g^{(2)}}{\partial \xi} + \frac{1}{\rho_0 V_0}\frac{\partial \rho_i^{(2)}}{\partial \xi} - \frac{gH}{V_0}\frac{\partial \alpha^{(2)}}{\partial \xi} &= -\frac{\partial u_g^{(1)}}{\partial \tau} - \frac{u_g^{(1)}}{V_0}\frac{\partial u_g^{(1)}}{\partial \xi} \\
&\quad - \frac{1}{3}\frac{\alpha^{(0)}H^2}{V_0}\left(\frac{u_g^{(0)}}{V_0} - 1\right)^2 \frac{\partial^3 \alpha^{(1)}}{\partial \xi^3},
\end{aligned} \qquad [30]
$$

$$
\begin{aligned}
\left(\frac{u_l^{(0)}}{V_0} - 1\right)\frac{\partial u_l^{(2)}}{\partial \xi} + \frac{1}{\rho_l V_0}\frac{\partial p_i^{(2)}}{\partial \xi} - \frac{gH}{V_0}\frac{\partial \alpha^{(2)}}{\partial \xi} &= -\frac{\partial u_l^{(1)}}{\partial \tau} - \frac{u_l^{(1)}}{V_0}\frac{\partial u_l^{(1)}}{\partial \xi} \\
&\quad + \frac{1}{3}\frac{(1-\alpha^{(0)})}{V_0}H^2\left(\frac{u_l^{(0)}}{V_0} - 1\right)^2 \frac{\partial^3 \alpha^{(1)}}{\partial \xi^3}.
\end{aligned} \qquad [31]
$$

The differential equations [24]–[27] may be integrated and the boundary conditions [23] applied to yield

$$\left(\frac{u_g^{(0)}}{V_0} - 1\right)\alpha^{(1)} = -\frac{\alpha^{(0)}}{V_0}u_g^{(1)}, \qquad [32]$$

$$\left(\frac{u_l^{(0)}}{V_0} - 1\right)\alpha^{(1)} = \frac{1-\alpha^{(0)}}{V_0}u_l^{(1)}, \qquad [33]$$

$$\left(\frac{u_g^{(0)}}{V_0} - 1\right)u_g^{(1)} + \frac{1}{\rho_g V_0}p_i^{(1)} - \frac{gH}{V_0}\alpha^{(1)} = 0, \qquad [34]$$

$$\left(\frac{u_l^{(0)}}{V_0} - 1\right)u_l^{(1)} + \frac{1}{\rho_l V_0}p_i^{(1)} - \frac{gH}{V_0}\alpha^{(1)} = 0. \qquad [35]$$

The quantities superscripted with 1 may be eliminated to give an equation for the phase velocity V_0

$$(u_g^{(0)} - V_0)^2\rho_g(1-\alpha^{(0)}) + gH\alpha^{(0)}(1-\alpha^{(0)})(\rho_g - \rho_l) + \rho_l\alpha^{(0)}(u_l^{(0)} - V_0)^2 = 0. \qquad [36]$$

This is identical to the relationship obtained in Part I by a linear dispersion analysis of the conservation equations. The phase velocity V_0 may be obtained by solving [36] as

$$V_0 = \frac{(1-\alpha^{(0)})u_g^{(0)}\rho_g + \alpha^{(0)}u_l^{(0)}\rho_l}{\alpha^{(0)}\rho_l + (1-\alpha^{(0)})\rho_g} \pm \left[\frac{-(u_g^{(0)} - u_l^{(0)})^2}{\alpha^{(0)}/\rho_g + (1-\alpha^{(0)})/\rho_l} + (\rho_l - \rho_g)gH\right]^{1/2} \times \left[\frac{\rho_g}{\alpha^{(0)}} + \frac{\rho_l}{1-\alpha^{(0)}}\right]^{1/2}, \qquad [37]$$

and is wholly real when

$$(\rho_l - \rho_g)gH\left[\frac{\alpha^{(0)}}{\rho_g} + \frac{1-\alpha^{(0)}}{\rho_l}\right] \geq (u_g - u_l)^2. \qquad [38]$$

If the variables superscripted (2) in [28]–[31] are eliminated using [32]–[34], then we obtain an expression of the form

$$A\frac{\partial^3\alpha^{(1)}}{\partial\xi^3} + B\alpha^{(1)}\frac{\partial\alpha^{(1)}}{\partial\xi} + C\frac{\partial\alpha^{(1)}}{\partial\tau} + D\frac{\partial\alpha^{(2)}}{\partial\xi} = 0. \qquad [39]$$

Now

$$D = \rho_l\alpha[u_l - V_0]^2 + \rho_g(1-\alpha^{(0)})[u_g^{(0)} - V_0]^2 + gH\alpha^{(0)}(1-\alpha^0)(\rho_g - \rho_l) = 0$$

from [36]. Therefore, we obtain the Korteweg–deVries equation:

$$A\frac{\partial^3\alpha^{(1)}}{\partial\xi^3} + B\alpha^{(1)}\frac{\partial\alpha^{(1)}}{\partial\xi} + C\frac{\partial\alpha^{(1)}}{\partial\tau} = 0, \qquad [40]$$

where

$$A = -\frac{1}{3V_0^3}[\rho_g\alpha^{(0)}H^2(u_g^{(0)} - V_0)^2 + \rho_l(1-\alpha^{(0)})H^2(u_l - V_0)^2],$$

$$B = \frac{3}{V_0}\left[\frac{\rho_g(u_g - V_0)^2}{\alpha^{(0)2}} - \frac{\rho_l(u_l^{(0)} - V_0)^2}{(1-\alpha^{(0)})^2}\right],$$

$$C = -2\left[\frac{\rho_g(u_g^{(0)} - V_0)}{\alpha^{(0)}} - \frac{\rho_l(u_l^{(0)} - V_0)}{(1-\alpha^{(0)})}\right].$$

Note that this is identical to the Korteweg–deVries equation obtained in Part I for incompressible, inviscid stratified flow by an analysis of the local instantaneous two-dimensional conservation equations and boundary conditions. Similar equations are obtained for $u_k^{(1)}$ and $p_i^{(1)}$ as is evident from [32] to [35].

5. CONCLUSIONS

The averaged conservation equations for stratified two-phase flow in a duct have been shown to involve higher order derivatives of the void fraction when the difference between the phase average and interfacial pressures are modelled with the help of the transverse momentum equation. An exact solution for the pressure difference cannot be obtained with the averaged equations, so the simplest possible model that satisfies the boundary conditions and kinematic conditions at the interface has been used. It is shown that this results in a linear dispersion relationship that is identical to the one for long waves obtained by analysing the local instantaneous equations. Furthermore, finite amplitude waves are found to be described by the identical Korteweg–deVries equation as obtained in Part I using the local instantaneous formulation. This enhances confidence in the approximations made in modelling the interfacial phase average pressure difference in the averaged equations. We conclude that:

- To properly model propagation, dispersion and damping of waves in separated two-phase flow, it is necessary to consider higher order derivative terms in the phasic conservation equations.

● These higher order derivative terms, at least in the case considered, arise naturally when the interfacial and phase average pressures are better modelled than done previously in Part I.

● The transverse momentum conservation equation should be considered in modelling the pressures; simple approximations appear adequate provided they satisfy the boundary and kinematic conditions for the transverse velocity.

● The physical reason that the higher order dispersive terms arise is related to the fact that the shape and motion of the interface are part of the problem being solved and cannot be specified arbitrarily; therefore, the kinematic condition at the interface must be satisfied.

It is expected that higher order dispersive terms of the type obtained in this paper will also be important in the study of waves in other two-phase flow regimes. The next step, however, would be to incorporate dissipation in the analysis and determine whether it occurs as a higher derivative term or as an algebraic term. Though this is speculation at this stage, a good model equation for wave phenomena in two-phase flow may be of the form of the Korteweg–deVries Burges equation derived by Johnson (1969) for blood flow through elastic tubes.

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